UAHEP 961
January 1996

Bosonic Description of Spinning Strings in $2+1$ Dimensions

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ABSTRACT

We write down a general action principle for spinning strings in $2+1$ dimensional space-time *without introducing Grassmann variables*. The action is written solely in terms of coordinates taking values in the $2+1$ Poincaré group, and it has the usual string symmetries, i.e. it is invariant under a) diffeomorphisms of the world sheet and b) Poincaré transformations. The system can be generalized to an arbitrary number of space-time dimensions, and also to spinning membranes and p-branes.

It is well known that the classical spin of relativistic particles can be described by using either classical or pseudoclasssical variables.[1] Here we show that the same holds for spinning strings. Since the psuedoclassical description of spinning strings is well known, we obviously are implying that there exist descriptions of spinning strings solely in terms of classical variables. For reasons of simplicity, we shall examine strings in $2+1$ space-time only. (The string action has a particularly elegant form in $2+1$ dimensions due to the existence of a nondegenerate scalar product on the Poincaré algebra $\underline{ISO(2,1)}$.[2]) Our system contains the general description of (spinless) strings due to Balachandran, Lizzi and Sparano[3] as a special case.[†] Furthermore, it can be generalized to an arbitrary number of space-time dimensions, and also to spinning membranes and p-branes. We shall discuss such generalizations in a later article.

Analogous to the bosonic formulation of a spinning particle in $2+1$ dimensions (cf. [4]), we write the spinning string action on the $2+1$ dimensional Poincaré group manifold $ISO(2,1)$. That is, the string variables are maps from the two dimensional world sheet to $ISO(2,1)$. We shall express the action for strings in $2+1$ dimensional Minkowski space in terms of these variables and their derivatives. It will be seen to be invariant under a) diffeomorphisms of the world sheet and b) global Poincaré transformations. The strings can be classified in terms of $ISO(2,1)$ orbits, and we shall find that certain orbits correspond to strings with a nonvanishing spin current. Lastly, we shall show how to embed such strings in curved space-time, the resulting action being invariant under a), and now b') local Poincaré transformations.

We denote the string variables by g. g can be decomposed into an $SO(2,1)$ matrix $\Lambda = \{\Lambda^i{}_j \ , \ i,j = 0,1,2\}$ and an $SO(2,1)$ vector $x = \{x^i \ , \ i = 0,1,2\}$. The latter will denote the Minkowski coordinates of the string. Under the left action of the Poincaré group,

[†]Spinning strings were also considered in [3] using a Wess-Zumino term. Here we shall show that there are more possibilities for including spin.

$g = (\Lambda, x)$ transforms according to the semidirect product rule:

$$g \to h \circ g = (\theta, y) \circ (\Lambda, x) = (\theta\Lambda, \theta x + y) \ . \tag{1}$$

We let t_i and u_i, $i = 0, 1, 2$ denote a basis for the Lie-algebra $\underline{ISO(2,1)}$. For their commutation relations we can take

$$[t_i, t_j] = \epsilon_{ijk} t^k \ , \quad [t_i, u_j] = \epsilon_{ijk} u^k \ , \quad [u_i, u_j] = 0 \ , \tag{2}$$

where we raise and lower indices using the Minkowski metric $[\eta_{ij}] =$diag$(-1, 1, 1)$, and we define the totally antisymmetric tensor ϵ_{ijk} such that $\epsilon^{012} = 1$.

A left invariant Maurer-Cartan form can be expanded in this basis as follows:

$$g^{-1} dg = \frac{1}{2} \epsilon^{ijk} (\Lambda^{-1} d\Lambda)_{ij} t_k + (\Lambda^{-1} dx)^i u_i \ . \tag{3}$$

It is easy to check that $g^{-1}dg$ is unchanged under the left action of the Poincaré group (1). Under the right action of the Poincaré group,

$$g \to g \circ h^{-1} = (\Lambda, x) \circ (\theta^{-1}, -\theta^{-1} y) = (\Lambda\theta^{-1}, x - \Lambda\theta^{-1} y) \ , \tag{4}$$

and consequently the Maurer-Cartan form transforms as

$$g^{-1} dg \to {}^h[g^{-1} dg] = \frac{1}{2} \epsilon^{ijk} (\Lambda^{-1} d\Lambda)_{ij} \ {}^h[t_k] + (\Lambda^{-1} dx)^i \ {}^h[u_i] \ , \tag{5}$$

where

$${}^h[t_i] = \theta^j{}_i t_j + \epsilon^{jk\ell} \theta_{ki} y_j u_\ell \ , \quad {}^h[u_i] = \theta^j{}_i u_j \ . \tag{6}$$

Here ${}^h[t_i]$ and ${}^h[u_i]$ denote basis vectors which are transformed under the adjoint action by $h \in ISO(2,1)$. They are given explicitly for $h = (\theta, y)$. These equations can be utilized to define the adjoint action $v \to {}^h[v]$ by $h \in ISO(2,1)$ on any vector $v = \alpha^i t_i + \beta^i u_i$ in $\underline{ISO(2,1)}$.

Two scalar products exist on $\underline{ISO(2,1)}$ which are invariant under the above defined adjoint action. We denote them by $<\ ,\ >$ and $(\ ,\ )$. The former satisfies

$$<t_i,u_j>=\eta_{ij}\ ,\quad <t_i,t_j>=<u_i,u_j>=0\ , \tag{7}$$

and is nondegenerate. The latter satisfies

$$(t_i,t_j)=\eta_{ij}\ ,\quad (u_i,t_j)=(u_i,u_j)=0\ , \tag{8}$$

and is degenerate. Thus for any two vectors v and v' in $\underline{ISO(2,1)}$ we have that $<{}^h[v],{}^h[v']>$ $=<v,v'>$ and $({}^h[v],{}^h[v'])=(v,v')$.

The nondegenerate scalar product $<\ ,\ >$ was utilized previously in writing down the action for a relativistic spinning particle.[4] The expression for the action is linear in the Maurer-Cartan form and it is therefore invariant under (left) Poincaré transformations, as well as diffeomorphisms of the particle world line. The action is just

$$S_{particle}=\int <K,g^{-1}dg>\ , \tag{9}$$

where here g is a function of the world line and K is a constant vector in $\underline{ISO(2,1)}$. The direction of K in $\underline{ISO(2,1)}$ determines whether the particle is massive, massless or tachyonic, and whether it is spinning or spinless. Actually, for this purpose, it is sufficient to specify the $ISO(2,1)$ orbit on which K lies. This is because both $K=K_0$ and $K={}^h[K_0]$, $h\in ISO(2,1)$, lead to the same classical equations of motion. This follows from the invariance property of the scalar product $<\ ,\ >$,

$$<K,g^{-1}dg>=<{}^h[K],{}^h[g^{-1}dg]>=<{}^h[K],g'^{-1}dg'>\ , \tag{10}$$

where $g'=g\circ h^{-1}$. Thus the action is invariant under $K\rightarrow {}^h[K]$ and the change of coordinates $g\rightarrow g'$. Now to specify the $ISO(2,1)$ orbit we can use the two invariants $<K,K>$ and (K,K) . Spin is associated with the former invariant, and we shall find an analogous result for strings as well.

For the case $K = mt_0 - \kappa u_0$, we end up with the known[4] bosonic description of a massive spinning particle. That choice corresponds to both invariants being nonvanishing: $< K, K >= m\kappa$ and $(K, K) = -m^2$. For this case the action can be expressed according to

$$< K, g^{-1}dg >= m\Lambda^i{}_0 dx_i + \kappa(\Lambda^{-1}d\Lambda)_{12} \ . \tag{11}$$

The equations of motion for the particle are easy to obtain. Transforming g according to: $g \rightarrow (1+\epsilon) \circ g$, where ϵ is an infinitesimal element of $\underline{ISO(2,1)}$, induces the following variation of the Maurer-Cartan form:

$$\delta(g^{-1}dg) = {}^{g^{-1}}[d\epsilon] \ . \tag{12}$$

Then

$$\delta S_{particle} = \int < K, {}^{g^{-1}}[d\epsilon] >= \int < {}^{g}[K], d\epsilon >= -\int < d({}^{g}[K]), \epsilon > , \tag{13}$$

where we have used the invariance property of the scalar product. The equations of motion thus state that ${}^{g}[K] = p^i t_i - j^i u_i$ are constants of the motion. Upon once again choosing $K = mt_0 - \kappa u_0$, we get the following expressions for these constants:

$$p^i = m\Lambda^i{}_0 \ , \quad j^i = m\epsilon^{ijk}x_j\Lambda_{k0} + \kappa\Lambda^i{}_0 \ . \tag{14}$$

The former can be identified with the momenta of the particle, while the latter can be identified with the angular momenta, the first term being the orbital angular momenta and the second being the spin. The spin is thus proportional to κ which is nonvanishing when $< K, K >$ is.

We now apply an analogous procedure to the description of spinning strings. Once again we shall express the action in terms of the Maurer-Cartan form and consequently it will be invariant under (left) Poincaré transformations. The action should be quadratic in $g^{-1}dg$ in order for it to also be invariant under diffeomorphisms of the world sheet. We then take the

tensor product of two such Maurer-Cartan forms and write

$$S_{string} = \int < \mathcal{K}, g^{-1}dg \otimes g^{-1}dg > \ . \tag{15}$$

Now $\mathcal{K}$ is a constant tensor with values in $\underline{ISO(2,1)} \otimes \underline{ISO(2,1)}$. Analogous to what happens for the particle action, it is sufficient to specify the $ISO(2,1)$ orbit on which $\mathcal{K}$ lies. This is because both $\mathcal{K} = \mathcal{K}_0$ and $\mathcal{K} = {}^h[\mathcal{K}_0]$, $h \in ISO(2,1)$, lead to the same classical equations of motion, which is once again due to the invariance property of the scalar product $< \, , \, >$,

$$< \mathcal{K}, g^{-1}dg \otimes g^{-1}dg > = < {}^h[\mathcal{K}], {}^h[g^{-1}dg] \otimes {}^h[g^{-1}dg] > = < {}^h[\mathcal{K}], g'^{-1}dg' \otimes g'^{-1}dg' > \ , \tag{16}$$

where $g' = g \circ h^{-1}$. Thus the action is invariant under $\mathcal{K} \to {}^h[\mathcal{K}]$ and the change of coordinates $g \to g'$. Now to specify the $ISO(2,1)$ orbit we can use the two invariants $< \mathcal{K}, \mathcal{K} >$ and $(\mathcal{K}, \mathcal{K})$.

The string action (15) has already been studied[3] for the case $\mathcal{K} = \epsilon^{ijk} n_k \ t_i \otimes t_j$. For that choice

$$< \mathcal{K}, g^{-1}dg \otimes g^{-1}dg > = \epsilon_{ijk} (\Lambda n)^i \ dx^j \ dx^k \ . \tag{17}$$

It remains to specify the constant vector n_i. For n_i space-like, light-like or time-like one recovers the Nambu string, the null string[3],[5] or the tachyonic string, respectively. Thus to get the Nambu string we may write, $n_i = \frac{1}{4\pi\alpha'} \ \delta_{i2}$. To see how this works we may extremize the action with repect to the variations of Λ: $\delta\Lambda_{ij} = \epsilon_{ik\ell} \Lambda^k{}_j \zeta^\ell$, ζ^ℓ being infinitesimal. This leads to the equations of motion

$$\epsilon^{ijk} \Lambda_{j2} V_k = 0 \ , \quad V_k = \epsilon_{ijk} dx^i \ dx^j \ . \tag{18}$$

Then V_i is parallel to Λ_{i2}. Upon fixing the normalization, we get $\Lambda_{i2} = V_i / \sqrt{V_j V^j}$. Substituting this result back into the integrand (17) yields the usual form for the Nambu action,

$$S_{Nambu} = \frac{1}{4\pi\alpha'} \int \sqrt{V_j V^j} = \frac{1}{2\pi\alpha'} \int d^2\sigma \ \sqrt{(\partial_0 \vec{x})^2 (\partial_1 \vec{x})^2 - (\partial_0 \vec{x} \cdot \partial_1 \vec{x})^2} \ , \tag{19}$$

where $\sigma = (\sigma_0, \sigma_1)$ parametrize the world sheet and $\partial_0 = \frac{\partial}{\partial\sigma_0}$ and $\partial_1 = \frac{\partial}{\partial\sigma_1}$.

We now return to the general form for the string action (15). The equations of motion are obtained in identical fashion as was done for the particle. Transforming g according to: $g \rightarrow (1+\epsilon)\circ g$, where ϵ is an infinitesimal element of $\underline{ISO(2,1)}$, once again induces the variation (12) of the Maurer-Cartan form. Then

$$\delta S_{string} = 2\int < \mathcal{K}, {}^{g^{-1}}[d\epsilon] \otimes g^{-1}dg >= 2\int < {}^{g}[\mathcal{K}], d\epsilon \otimes dgg^{-1} > , \tag{20}$$

where we have used the invariance property of the scalar product and the identity

$${}^{g^{-1}}[dgg^{-1}] = g^{-1}dg \; . \tag{21}$$

Upon integrating by parts we arrive at the equations of motion

$$d \; < {}^{g}[\mathcal{K}], T_A \otimes dgg^{-1} >= 0 \; , \tag{22}$$

where the T_A's denote the generators of $ISO(2,1)$. These equations state that there are six conserved currents. For $T_A = u_i$, we have

$$\partial_\alpha P_i^\alpha = 0 \; , \qquad P_i^\alpha = \epsilon^{\alpha\beta} < {}^{g}[\mathcal{K}], u_i \otimes \partial_\beta g g^{-1} > , \tag{23}$$

which we identify with the momentum current conservation. (Here $\alpha, \beta, ...$ denote world sheet indices.) For $T_A = t_i$, we have

$$\partial_\alpha J_i^\alpha = 0 \; , \qquad J_i^\alpha = \epsilon^{\alpha\beta} < {}^{g}[\mathcal{K}], t_i \otimes \partial_\beta g g^{-1} > , \tag{24}$$

which we identify with the angular moment current conservation.

We now examine the conserved currents P_i^α and J_i^α for several cases.

Case 1: $\mathcal{K} = \mathcal{K}_1 = \epsilon^{ijk} n_k \; t_i \otimes t_j$. This is the case we considered earlier which contains the Nambu string, as well as the null and tachyonic strings. It corresponds to the $ISO(2,1)$ orbit with $< \mathcal{K}, \mathcal{K} >= 0$ and $(\mathcal{K}, \mathcal{K}) = -2n_i n^i$. Here we get that

$${}^{g}[\mathcal{K}_1] \;\; = \;\; \epsilon^{ijk}(\Lambda n)_k \; t_i \otimes t_j - \epsilon^{ijk}(\Lambda n)_\ell x^\ell x_k \; u_i \otimes u_j$$

$$+ (\Lambda n)_\ell x^\ell \; (t_i \otimes u^i - u_i \otimes t^i) + (\Lambda n)^i x^j \; (u_i \otimes t_j - t_j \otimes u_i) \; . \tag{25}$$

Using this and the expression for the right invariant Maurer-Cartan form

$$dgg^{-1} = \frac{1}{2}\epsilon^{ijk}(d\Lambda\Lambda^{-1})_{ij}t_k + [dx^i - (d\Lambda\Lambda^{-1}x)^i]u_i \; , \tag{26}$$

we compute the following currents:

$$\begin{aligned} P_i^\alpha = P^\alpha_{(1)i} \;\; &= \;\; \epsilon^{\alpha\beta}\Big(\frac{1}{2}\epsilon_{mjk}(\partial_\beta\Lambda\Lambda^{-1})^{jk}[\delta_i^m(\Lambda n)_\ell x^\ell - (\Lambda n)^m x_i] \\ &\qquad + \epsilon_{ijk}[\partial_\beta x^j - (\partial_\beta\Lambda\Lambda^{-1}x)^j](\Lambda n)^k\Big) \; , \end{aligned} \tag{27}$$

$$\begin{aligned} J_i^\alpha = J^\alpha_{(1)i} \;\; &= \;\; -\epsilon^{\alpha\beta}\Big((\partial_\beta\Lambda\Lambda^{-1})_i{}^j(\Lambda n)_\ell x^\ell x_j \\ &\qquad + [\partial_\beta x^j - (\partial_\beta\Lambda\Lambda^{-1}x)_j][\delta_i^j(\Lambda n)_\ell x^\ell - (\Lambda n)_i x^j]\Big) \; . \end{aligned} \tag{28}$$

These two currents can be shown to be related by

$$J^\alpha_{(1)i} - \epsilon_{ijk}x^j P^{k\alpha}_{(1)} = 0 \; . \tag{29}$$

We therefore argue that for such strings the angular momentum current consists only of an orbital term, and that no spin is present. This is a well known result for the Nambu string.

Case 2: $\mathcal{K} = \mathcal{K}_2 = \frac{1}{2}\chi^{ij}\;(u_i \otimes t_j - t_j \otimes u_i)$. Since the action (15) is antisymmetric with respect to the exchange of the two vector spaces, this is the most general ansatz for the tensor $\mathcal{K}$ which is linear in u_i and in t_j. It corresponds to the $ISO(2,1)$ orbit with $<\mathcal{K},\mathcal{K}>= \chi_{ij}\chi^{ij}$ and $(\mathcal{K},\mathcal{K}) = 0$. From it we get

$$\begin{aligned} {}^g[\mathcal{K}_2] \;\; &= \;\; \frac{1}{2}(\Lambda\chi\Lambda^{-1})^{ij}\;(u_i \otimes t_j - t_j \otimes u_i) \\ &\qquad + \frac{1}{2}[\mathrm{tr}\chi \; x_k - (\Lambda\chi\Lambda^{-1})^\ell{}_k x_\ell]\;\epsilon^{ijk}\;u_i \otimes u_j \; . \end{aligned} \tag{30}$$

We now obtain the following currents:

$$P_i^\alpha \;\; = \;\; P^\alpha_{(2)i} = -\frac{1}{4}\epsilon^{\alpha\beta}\epsilon_{jk\ell}(\Lambda^{-1}\partial_\beta\Lambda)^{jk}(\chi\Lambda^{-1})^\ell{}_i \; , \tag{31}$$

$$J_i^\alpha \;=\; J^\alpha_{(2)i} = \frac{1}{2}\epsilon^{\alpha\beta}\Big([\Lambda\chi\partial_\beta(\Lambda^{-1}x)]_i - [\mathrm{tr}\chi\ (\partial_\beta\Lambda\Lambda^{-1})_{ji} + (\Lambda\chi\partial_\beta\Lambda^{-1})_{ji}]x^j\Big)\ . \tag{32}$$

Now the analogue of eq. (29) is no longer true, i.e.

$$S_i^\alpha = J^\alpha_{(2)i} - \epsilon_{ijk}x^j P^{k\alpha}_{(2)} \neq 0\ . \tag{33}$$

We then conclude that a spin current is present in this case.

Case 3: $\mathcal{K} = \mathcal{K}_3 = \epsilon^{ijk}\ \nu_k\ u_i \otimes u_j$. Here both invariants vanish, $< \mathcal{K}, \mathcal{K} >= 0$ and $(\mathcal{K}, \mathcal{K}) = 0$. The currents P_i^α and J_i^α are trivially conserved in this case. This is because the integrand in (15) can be expressed as an exact two form on $ISO(2, 1)$:

$$< \mathcal{K}, g^{-1}dg \otimes g^{-1}dg >= \frac{1}{2}\epsilon_{ijk}(\Lambda^{-1}d\Lambda)^{ij}(\Lambda^{-1}d\Lambda\ \nu)^k = -\ d\ \epsilon_{ijk}\ \nu^k(\Lambda^{-1}d\Lambda)^{ij}\ . \tag{34}$$

Although this term does not contribute to the classical equations of motion, it can affect the quantum dynamics. Furthermore, it is known to be associated with the θ-vacua of string theory.[3],[6]

Case 4: $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$. This defines the most general classical system with the string action given by (15). It therefore contains the case of the Nambu string. Now both invariants can be nonzero, $< \mathcal{K}, \mathcal{K} >= \chi_{ij}\chi^{ij}$ and $(\mathcal{K}, \mathcal{K}) = -2n_i n^i$. The conserved currents are now given by:

$$P_i^\alpha \;=\; P^\alpha_{(1)i} + P^\alpha_{(2)i}\ , \tag{35}$$

$$J_i^\alpha \;=\; J^\alpha_{(1)i} + J^\alpha_{(2)i}\ . \tag{36}$$

For such strings, we can identify both an orbital and a spin angular momentum current, i.e. $J_i^\alpha = L_i^\alpha + S_i^\alpha$. The spin current S_i^α is defined in eq. (33), while the orbital angular momentum L_i^α is given by

$$L_i^\alpha = J^\alpha_{(1)i} + \epsilon_{ijk}x^j P^{k\alpha}_{(2)}\ . \tag{37}$$

From the above discussion we conclude that a spin current is present for the case of $ISO(2,1)$ orbits with $< \mathcal{K}, \mathcal{K} > \neq 0$. If we include a Wess-Zumino term as is done in ref. [3] an additional term of the form $\epsilon_{ijk}\epsilon^{\alpha\beta}(\Lambda^{-1}\partial_\beta\Lambda)^{jk}$ contributes to the angular momentum current.

It is easy to embed our spinning strings in curved space-time. Now the action should be invariant under local Poincaré transformations,

$$g \to h_L \circ g \ , \tag{38}$$

where like g, h_L are functions on the two dimensional world sheet, taking values in $ISO(2,1)$. We recall that the action (15) was instead invariant under global Poincaré transformations. To elevate it to a local invariance, we replace $g^{-1}dg$ by

$$g^{-1}Dg = g^{-1}dg + {}^{g^{-1}}[A] \ , \tag{39}$$

where $A = \omega^i t_i + e^i u_i$ are the connection one-forms for $ISO(2,1)$ evaluated on the string world sheet. Under Poincaré gauge transformations (38),

$$A \to {}^{h_L}[A] - dh_L {h_L}^{-1} \ , \tag{40}$$

and as a result Dgg^{-1} is invariant. Then

$$\mathcal{S}_{string} = \int < \mathcal{K}, g^{-1}Dg \otimes g^{-1}Dg > \ . \tag{41}$$

is gauge invariant, and hence gives the string action in curved space-time.

The equations of motion obtained by varying g now state that the momentum and angular momentum currents are covariantly conserved. To see this we can again use (12) along with $\delta({}^{g^{-1}}[A]) = {}^{g^{-1}}[A, \epsilon]$. Then

$$\delta\mathcal{S}_{string} = 2\int < \mathcal{K}, {}^{g^{-1}}[D\epsilon] \otimes g^{-1}Dg >= 2\int < {}^{g}[\mathcal{K}], D\epsilon \otimes Dgg^{-1} > \ , \tag{42}$$

where we have used the invariance property of the scalar product, $D\epsilon = d\epsilon + [A, \epsilon]$ and the identity ${}^{g^{-1}}[Dgg^{-1}] = g^{-1}Dg$. Upon integrating by parts we now arrive at the equations of motion

$$d \ < {}^{g}[\mathcal{K}], T_A \otimes Dgg^{-1} > - < {}^{g}[\mathcal{K}], [A \otimes 1\!\!1, T_A \otimes Dgg^{-1}] >= 0 \ , \tag{43}$$

where the T_A's once again denote the generators of $ISO(2,1)$. We then get the following generalizations of (23) and (24):

$$\partial_\alpha \mathcal{P}^\alpha_i + \epsilon_{ijk}\omega^j_\alpha \mathcal{P}^{\alpha k} \ = \ 0 \ , \qquad \mathcal{P}^\alpha_i = \epsilon^{\alpha\beta} < {}^{g}[\mathcal{K}], u_i \otimes \mathcal{D}_\beta gg^{-1} > \ , \tag{44}$$

$$\partial_\alpha \mathcal{J}^\alpha_i + \epsilon_{ijk}(\omega^j_\alpha \mathcal{J}^{\alpha k} + e^j_\alpha \mathcal{P}^{\alpha k}) \ = \ 0 \ , \qquad \mathcal{J}^\alpha_i = \epsilon^{\alpha\beta} < {}^{g}[\mathcal{K}], t_i \otimes \mathcal{D}_\beta gg^{-1} > \ , \tag{45}$$

ω^j_α, e^j_α and $\mathcal{D}_\beta gg^{-1}$ denoting the world sheet components of the one forms ω^j, e^j and Dgg^{-1}, respectively. The currents play the role of sources for the $ISO(2,1)$ curvature. For this we can take the Chern-Simons action[2] for the fields. Then $\mathcal{P}^\alpha_i$ is a source for the $SO(2,1)$ curvature, while $\mathcal{J}^\alpha_i$ is a source for the torsion.

Acknowledgements: We are grateful for discussions with R. Casadio. We were supported in part by the Department of Energy, USA, under contract number DE-FG05-84ER40141.

References

[1] A. P. Balachandran, G. Marmo, B.-S. Skagerstam and A. Stern, Phys. Lett. **89B**, 199 (1980); *Gauge Symmetries and Fibre Bundles, Applications to Particle Dynamics*, Lecture Notes in Physics **188**, Springer-Verlag, Berlin, 1982.

[2] E. Witten, Nucl. Phys. **B311**, 46 (1988).

[3] A. P. Balachandran, F. Lizzi and G. Sparano, Nucl. Phys. **B263**, 608 (1986); **B277**, 359 (1986); A. P. Balachandran, F. Lizzi, R. D. Sorkin and G. Sparano, Nucl. Phys. **B287**, 508 (1987).

[4] B.-S. Skagerstam and A. Stern, Int. J. of Mod. Phys. **A5** , 1575 (1990).

[5] A. Schild, Phys. Rev. **D16**, 1722 (1977); F. Lizzi and G. Sparano, Phys. Lett. **B232**, 311 (1989); I. A. Bandos, D.P. Sorokin, M. Tonon, D.V. Volkov, Phys. Lett. **B319**, 445 (1993).

[6] P.O. Mazur and V.P. Nair, Nucl. Phys **B284** , 146 (1987); V.G.J. Rodgers, Mod. Phys. Lett. **A7**, 1001 (1992); J. Pawelczyk, Phys. Lett. **B311** , 98 (1993); R. Efraty, Phys. Lett. **B322**, 84 (1994).